Automorphic symmetry and AdS String novel integrable deformations

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arXiv:2003.04332, PRL
arXiv:2010.11231, JPhys-A
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arXiv:2112.10843, Springer
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Outline

- Introduction: Automorphic symmetries as a tool for finding new integrable structures
    - Quantum symmetries and the $\mathcal{B}$oost in continuous systems
    - Discrete integrability and symmetries
    - The correct discrete $\mathcal{B}$ counterpart
    - Generalised bottom-up approach: R
- Implementation for lattice integrability with $\mathbb{V} = \mathbb{C}^2$ and $\mathbb{V} = \mathbb{C}^4$ local spaces (*completeness*)

Outline

- Automorphism in AdS_n integrable backgrounds (General method)
- Novel integrable classes arising in string backgrounds: Spin chain picture of 6vB/8vB
- B-class integrable properties: $\mathfrak{B}^{\mathcal{X}}$, $\mathbb{C}_{AdS_n}$, *Free fermion*
- Further directions in $AdS_3 \times S^3 \times \mathcal{M}^4$ (*ABA*/*TBA*, *Excitations* and more)

Introduction

Definition. Generically a dynamical system is *integrable*, when it possesses *sufficient* number of motion integrals and its dynamics can be described by $N < D$ dof, where D is the dimension of the underlying phase space.

Conventionally, it acquires **three characteristics**:

- saturating set of conservation laws
- algebraic invariants
- existence of explicit functional solution

Theorem [Liouville]. If the system is Liouville integrable then associated equations of motion are quadrature solvable.

Classically integrable

Apart from present conserved quantities, there also exist **sufficient integrable constraints**.

Provided some matrix algebra $\mathfrak{g}$ along with Jacobi identity to hold on Poisson bracket, one can obtain a constraint on embedded element $\exists\, r_{12} \in \mathfrak{g} \otimes \mathfrak{g}$, **C**lassical **Y**ang-**B**axter **E**quation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \begin{cases} 0, \text{ CYBE} \\ \mathcal{C}, \text{ modified-CYBE} \end{cases} \qquad r_{21} = \mathcal{P} r_{12}$$

r being a *constant classical r-matrix*. With mCYBE to play a central role in σ-model integrability.

Quantum Integrable System

The quantum integrability has analogous characteristics, i.e. **quantum mechanical** or **quantum field theoretic** systems contain sufficient number of **motion integrals** (predominantly infinite set) in one spatial dimension ($[\mathfrak{Q}, \mathbb{H}] = 0$).

In particular such **(1+1)-dimensional quantum models**, lead to a very rich integrable structure equipped with variety of important physical properties, e.g. **sine-Gordon**

$$\Box\phi + m^2 \sin\phi = 0$$

On the other hand field theories with interaction are accompanied with several shortcomings. As known, one of the central flaws, are **singularities** and one demands a **regularisation scheme** for it.

Discretisation

One way to regulate is to **discretise the space** or introduce a **lattice**.

That leads to a reduction of the field theoretic model in **finite volume** to a system with **finite dof**.

Such systems are qualified as magnetic chains of quantum spins, or **Q**uantum **S**pin **C**hains.

Spin chains appear to be a wide **universal class** of quantum integrable models. Moreover there exist *spatial continuous* limits to the associated 2-dim QFTs

$$\delta_{ij}/\Delta = \delta(x-y) \qquad\qquad x = N\Delta$$

with N spacings Δ, in the $\Delta \to 0, N \to \infty$ limit x becomes continuous variable.

From continuous setups it is known that a certain generating (conserved) observable, *e.g.* current $\mathcal{J}^{\mu}(x)$ possesses Lorentz transform

$$\mathcal{U}\mathcal{J}^{\mu}(x)\mathcal{U}^{-1} = \Lambda^{\mu}_{\nu}\,\mathcal{J}^{\nu}(\Lambda x)$$

where $\mathcal{U}\left[\Lambda(u)\right]$ and Λ^{ν}_{μ} is Lorentz operator with fugacity u. In the limit one acquires the boost

$$\mathcal{B} = \mathrm{d}\,_{u}\Lambda(u)\big|_{u=0} \qquad [\mathcal{B}, \mathcal{J}^{\mu}_{a}(x)] = \epsilon_{\alpha\beta}\,x^{\alpha}\partial^{\beta}\mathcal{J}^{\mu}_{a}(x) + \epsilon^{\mu\beta}\mathcal{J}^{\mu}_{a,\,\beta}$$

for the local charge it follows

$$[\mathcal{B}, Q_a] = 0 \qquad \text{with} \quad \mathcal{J}_a(\pm\infty) \equiv 0$$

Can be shown

$$e^{2\pi i\mathcal{B}}\hat{Q}_a e^{-2\pi i\mathcal{B}} = \hat{Q}_a - \frac{1}{2}\mathcal{C}Q_a$$

$\mathcal{C}$ is quadratic Casimir of some $\mathfrak{g}$ in the adjoint representation $(-\delta_{ab}\mathcal{C} = f_{acd}f_{cdb})$.

So that one can see that conserved charges transform under $\mathcal{B}$ as

$$[\mathcal{B}, Q_a] = 0 \qquad \left[\mathcal{B}, \hat{Q}_a\right] = -\frac{\mathcal{C}}{4\pi i} Q_a$$

where nonvanishing second commutator is obviously a quantum effect. In fact the boost intertwines **spacetime** and **internal** symmetries, which indicates that quantised $\mathcal{Y}$ manifests more than only an internal symmetry $\rightarrow$ stronger constraining of the symmetry invariants.

$$\text{Poincaré:} \qquad [\mathcal{P}_-, \mathcal{P}_+] = 0, \quad [\mathcal{B}, \mathcal{P}_\pm] = \pm\mathcal{P}_\pm$$

$$\text{Yangian supplement } \mathcal{Y}[\mathfrak{g}]: \quad \begin{cases} [\mathcal{P}_\pm, Q_a] = 0 \\ [\mathcal{B}, Q_a] = 0 \end{cases} \qquad \begin{array}{l} \left[\mathcal{P}_\pm,\ \hat{Q}_a\right] = 0 \\ \left[\mathcal{B},\ \hat{Q}\right] = -\mathfrak{C}\frac{\hbar}{4\pi i} Q_a \end{array}$$

where $\mathcal{P}_\pm$ are lightcone translations, ${}^{(n)}Q_a$ is level-n Yangian generators. We can notice that the boost $\mathcal{B}$ generates nontrivial internal symmetry extension [LeClair, Smirnov '91].

$$\rho\left[\mathcal{B}_{\tilde{u}} \otimes \mathcal{B}_{\tilde{u}}\left(\Delta^{\mathrm{op}}(a)\right)\right] S(u-v) = S(u-v)\,\rho\left[\mathcal{B}_{\tilde{u}} \otimes \mathcal{B}_{\tilde{u}}\left(\Delta(a)\right)\right]$$

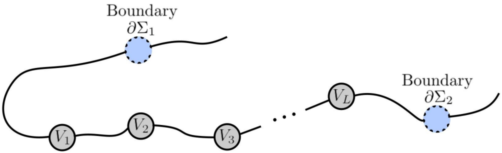
thus intertwining the Yangian evaluation modules.

Discretisation: ISC

An integrable spin chain can be characterised by the hierarchy of mutually commuting conserved quantities, *e.g.* **charge operators** $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_4$, $\ldots \mathbb{Q}_r$, where r denotes interaction range.

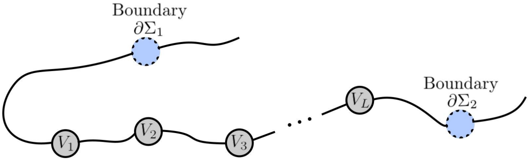
One considers local vector space $(n = 2s + 1)$ $\mathfrak{H} \simeq \mathbb{C}^n$, then spin-$s$ chain **configuration space** is given by the ***L*-fold tensor** product

$$\text{Complete Fock space:} \qquad H = \mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_L \equiv \bigotimes_{i=1}^{L} \mathfrak{H}_i \qquad \mathfrak{H}_i \simeq \mathfrak{H}$$



Important to address the **boundaries** of such quantum systems, since it can lead to very distinct physical description and properties. That could be seen from canonical **Heisenberg integrable class**.

Boundary Conditions



Distinct *boundary conditions* correspond to different types of integrable spin chains. $\partial\Sigma_i$ is a specific boundary and $V_i \in \mathbb{C}^n$ is n-dimensional local quantum space at site i.

One can classify 5 main boundary types for the spin chain

- *Infinite* spin chain: $-\infty \leftarrow \partial\Sigma_1$, $\partial\Sigma_2 \rightarrow +\infty$
- *Semi-infinite* spin chain: $\partial\Sigma_1 = V_0$, $\partial\Sigma_2 \rightarrow +\infty$
- *Open* spin chain: $\partial\Sigma_1 = V_0$, $\partial\Sigma_2 = V_{L+1}$
- *Closed* spin chain: $(0) \rightarrow (L+1)$, $\partial\Sigma_1 = \partial\Sigma_2 = V_{L+1}$
- *Cyclic* spin chain: $\partial\Sigma_1 = \partial\Sigma_2 = V_{L+1}$ along with shift symmetry $i \rightarrow i \pm 1$, where the total spin chain momentum of excitations $P = 0$.

Integrable Spin Chain $\mathbb{V} = \mathbb{C}^2$

A quantum discrete system with nearest-neighbour (two-body) ferromagnetic interaction is described by the Hamiltonian (*Heisenberg class*)

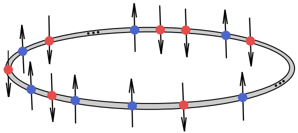
$$H = -\sum_{i=1}^{\tilde{L}} \left(J_i^x S_i^x S_{i+1}^x + J_i^y S_i^y S_{i+1}^y + J_i^z S_i^z S_{i+1}^z \right) + \text{1-body-interactions}$$

where J_i^k are real positive numbers at each site of the spin chain, S_k are spin operators. By considering various coupling $\boldsymbol{J_{x,y,z}}$ **interrelations** one finds different spin chains (XX, XY, XXX, XXZ, XYZ etc)

Shifts: $\mathbb{Q}_1 \equiv P$

NN-interacting charge: $\mathbb{Q}_2 \equiv H$

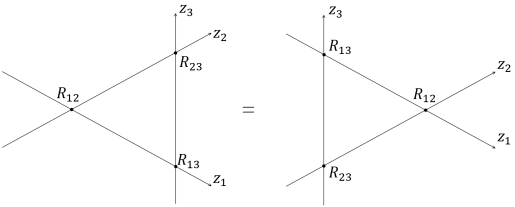
$$H = \sum_{n=1}^{L} \mathcal{H}_{n,n+1} \quad H_{L,L+1} \equiv H_{L,1}$$



Closed Heisenberg Chain Class

QYBE

But now with a **quantum integrable constraint**



2-dim scattering factorisation in integrable systems

Hence sufficient condition is described by **Quantum YBE** [Yang-Yang '66, Baxter '72] [Drinfeld '85]

$$R_{12}(z_1, z_2)\, R_{13}(z_1, z_3)\, R_{23}(z_2, z_3) = R_{23}(z_2, z_3)\, R_{13}(z_1, z_3)\, R_{12}(z_1, z_2)$$

Quantum R-matrix: $R_{ij} \in \mathrm{End}(\mathbb{V} \otimes \mathbb{V} \otimes \mathbb{V})$

$$R : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V} \qquad R_{12}(z, z) = P_{12}$$

$$R_{ab}(u,v)R_{ac}(u,w)R_{bc}(v,w) = R_{bc}(v,w)R_{ac}(u,w)R_{ab}(u,v)$$

In particular, one can consider **difference** (or additive) spectral form QYBE

$$R_{ab}(u,v) \to R_{ab}(u-v)$$

in addition from R-monodromy one can establish

$$\mathbb{Q}_2 = \sum_{n=1}^{L} R^{-1}_{n,n+1}(0)\frac{d}{du}R_{n,n+1} \equiv \sum_n \mathcal{Q}_{n,n+1}$$

with $\mathcal{Q}$ to denote local densities. In general we know that

$$\mathbb{Q}_r \equiv \sum_n \mathcal{Q}_{n,n+1,\dots,n+r-1} = -\frac{i}{(r-1)!}\frac{\mathrm{d}^{\,r-1}}{\mathrm{d}u^{r-1}}\log\left[t(u)\right]\Big|_{u=\frac{i}{2}}$$

however a method is required to construct all higher charges.

Emergence of Automorphic symmetry

One can note that it is useful to consider $R\mathcal{L}\mathcal{L}$ relation along the lines of QISM

$$R_{12}(v)\mathcal{L}_{10}(u+v)\mathcal{L}_{20}(u) = \mathcal{L}_{20}(u)\mathcal{L}_{10}(u+v)R_{12}(v)$$

By differential properties and multiplication of monodromic complements, one can obtain

$$i\prod_{j=1}^{k-1}\mathcal{L}_{0,j}\left[\mathcal{L}_{0,k}\mathcal{L}_{0,k+1},\mathcal{H}_{k,k+1}\right]\prod_{n=k+2}^{L}\mathcal{L}_{0,n} =$$

$$\prod_{j=1}^{k-1}\mathcal{L}_{0,j}\mathcal{L}_{0,k}\mathcal{L}'_{0,k+1}\prod_{n=k+2}^{L}\mathcal{L}_{0,n} - \prod_{j=1}^{k-1}\mathcal{L}_{0,j}\mathcal{L}'_{0,k}\mathcal{L}_{0,k+1}\prod_{n=k+2}^{L}\mathcal{L}_{0,n}$$

which after performing resummation and boundary limits

$$i\left[\sum_{k=-\infty}^{+\infty}k\mathcal{H}_{k,k+1}, T(u)\right] = \mathrm{d}_u T(u)$$

Automorphic condition

that would result in

$$i\,[\mathcal{B}, t] = \dot{t}$$

with $\dot{T} \equiv \mathrm{d}_u T(u)$ and transfer matrix $t(u)$. It can be noted, that it constitutes nothing but a discrete form of the field theoretic boost symmetry with a discretisation scheme $\int x \mathrm{d}\, x \mapsto \sum_k k$. One can straightforwardly identify the generating scheme

$$\mathcal{Q}_2 = \mathcal{U}^{-1}\,[\mathcal{B}, \mathcal{U}]$$

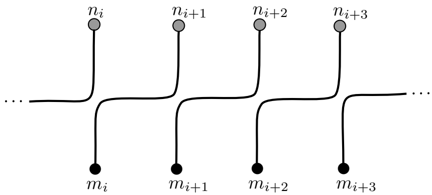
$$\mathcal{Q}_3 = \frac{i}{2}\left[\mathcal{U}^{-1}\,[\mathcal{B}, \mathcal{U}\mathcal{Q}_2] - \mathcal{Q}_2\mathcal{Q}_2\right] = \frac{i}{2}\Big[\underbrace{\left(\mathcal{U}^{-1}\,\mathcal{B}\,\mathcal{U} - \mathcal{Q}_2\right)}_{\mathcal{B}}\mathcal{Q}_2 - \mathcal{Q}_2\mathcal{B}\Big] = \frac{i}{2}\,[\mathcal{B}, \mathcal{Q}_2]$$

$$\mathcal{Q}_4 = \frac{i}{3}\,[\mathcal{B}, \mathcal{Q}_3]$$

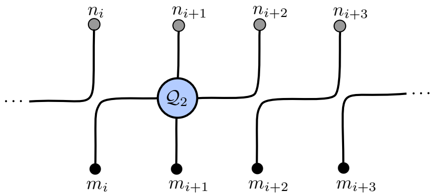
$$\dots$$

$$\mathcal{Q}_{r+1} = \frac{i}{r}\,[\mathcal{B}, \mathcal{Q}_r]$$

$$\dots$$



(a) $\mathcal{U}$



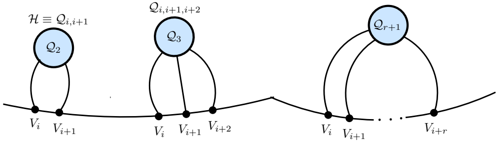
(b) $\mathcal{U}\,\mathbb{Q}_2$

First and second order of the expansion, corresponding to the shift operator $\mathcal{U}$ and product of the shift and Hamiltonian of the system

$\mathcal{B}$oost automorphism

From field theoretic perspective, **higher symmetries** [Tetelman '82] are reflected not only at the level of the **Poincaré algebra**, but would also admit infinite dimensional extension of **internal symmetries,** where the boost $\mathcal{B}[\,\cdot\,]$ manifests its **automorphic** nature

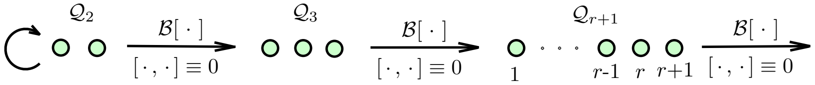
$$\mathcal{B}[\mathbb{Q}_2] \equiv \sum_{k=-\infty}^{\infty} k\mathcal{H}_{k,k+1} \quad \rightarrow \quad \mathbb{Q}_{r+1} \simeq [\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_r]$$



Range n local charges $\mathcal{Q}_n$

$\mathcal{B}$oost automorphism

Rigorously the boost operator is defined for **infinite length chains**. However analytic proof for $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$ **sufficiency***, i.e. $\forall$ levels of the integrable hierarchy **not present**. [Reshetikhin and Grabowski-Mathieu conjecture '95]



$\mathcal{B}$oost operator automorphically generates the conserved charge hierarchy.

The **Boost automorphism on the conserved charges** can be related to **Drinfeld automorphism on $\mathcal{Y}$-algebra**.

Ansatz for automorphism

In the case of $\mathbb{C}^2$ local space with the 2×2 spin σ^a embedding

$$\sigma_n^a = \mathbb{1} \otimes \ldots \underbrace{\otimes \, \sigma^a \otimes}_{\text{n}} \cdots \otimes \mathbb{1}$$

One can demand generic **Hamiltonian ansatz**

$$\mathcal{Q}_{ij} = A_{ab} \, \sigma^a \otimes \sigma^b$$

Resolution structure already emerges from the first commutator

$$[\mathbb{Q}_2, \mathbb{Q}_3] = \sum_{m,n} \mathcal{A}_{ab} \mathcal{A}_{efg} \left[\ldots \sigma_m^a \sigma_{m+1}^b \ldots , \ldots \sigma_n^e \sigma_{n+1}^f \sigma_{n+2}^g \ldots \right]$$
$$\equiv \mathcal{C}_{abef} \sum_m \ldots \sigma_m^a \sigma_{m+1}^b \sigma_{m+2}^e \sigma_{m+3}^f \ldots$$

and is hypothetically completely constraining. Generically $[\mathbb{Q}_r, \mathbb{Q}_s]$ commutator provides $\frac{1}{2}(3^{r+s-1} - 1)$ polynomial equations of degree $r + s - 2$.

To obtain generating solutions, reduction transformations must be applied to the full solution space

Transformations

- Choice of appropriate **normalisation** of $\mathcal{H}$ and addition of $\mathfrak{C}_i \cdot \mathbb{1}$
- **Local basis transform**

$$\tilde{\mathcal{Q}}_{\{i_1 \ldots i_L\}} = \Big(\bigotimes_L \mathcal{V} \Big) \mathcal{Q}_{\{i_1 \ldots i_L\}} \Big(\bigotimes_L \mathcal{V}^{-1} \Big)$$

  with unimodular $\mathcal{V}$.

- Set of **discrete transforms**

$$\begin{array}{ccc} R(u) & \leftrightarrow & \mathcal{H} \\ PR(u)P & \leftrightarrow & P\mathcal{H}P \\ R(u)^T & \leftrightarrow & P\mathcal{H}^T P \\ PR(u)^T P & \leftrightarrow & \mathcal{H}^T \end{array}$$

New classes: $\mathfrak{sl}_2$ deformed sector

One more new $\mathcal{H}$ has the form

$$\mathcal{H}_6 = \begin{pmatrix} a_1 & a_2 & a_2 & 0 \\ 0 & -a_1 & 2a_1 & -a_2 \\ 0 & 2a_1 & -a_1 & -a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix}$$

together with the unitary R matrix

$$R_6(u) = {\scriptstyle (1-a_1 u)(1+2a_1 u)} \begin{pmatrix} 1 & a_2 u & a_2 u & -a_2^2 u^2(2a_1 u+1) \\ 0 & \frac{2a_1 u}{2a_1 u+1} & \frac{1}{2a_1 u+1} & -a_2 u \\ 0 & \frac{1}{2a_1 u+1} & \frac{2a_1 u}{2a_1 u+1} & -a_2 u \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The six new classes develop **non-diagonalisability**, some are nilpotent and some exhibit distinct eigenspectrum – associated conserved charges develop **non-trivial Jordan blocks**.

Physical? Relation to conformal fishchain in the continuum? Temperley-Lieb or Hecke systems? Constitute **higher-parametric** deformations of Kulish-Stolin (deformed $\mathcal{Y}[\mathfrak{sl}_2]$) [Kulish, Stolin] [Alcaraz, Droz, Henkel, Rittenberg '93]

Graded space

We also extend our results to graded vector spaces $\mathbb{C}^{1|1}$

$$\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \quad \dim\mathbb{V}_0 = m \quad \dim\mathbb{V}_1 = n \quad \begin{cases} g(i) = 0,\ 1 \leq i \leq m \\ g(i) = 1, \text{ with } m \leq i \leq m+n \end{cases}$$

with grading $g(i)$, $i \in \{1, \dots, m+n\}$. For bosofermionic setting one obtains $i = 1, 2$ for $\mathcal{C}^{1|1}$, although integrable case restricts to **even sector**.

The *RTT* relation and graded ansatz constitute

$$T_a(u)T_b(u) = R^{-1}(u-v)T_a(u)T_b(u)R(u-v) \quad STr(\mathfrak{O}_A\mathfrak{O}_B) = (-1)^{\mathfrak{g}(A)+\mathfrak{g}(B)}STr(\mathfrak{O}_B\mathfrak{O}_A)$$

$$\begin{cases} Q_2 = \sum \mathcal{A}_{ijkl} E_{ij} \otimes_g E_{kl} \\ \mathcal{O}_I\mathcal{O}_{II} = \bigotimes_{i=1}^n \mathfrak{e}_{I,i} \bigotimes_{j=1}^n \mathfrak{e}_{II,j} \\ = \prod_{i=1}^{n-1}(-1)^{|\mathfrak{e}_{II,i}| \sum\limits_{j=i+1}^{n} |\mathfrak{e}_{I,j}|} \bigotimes_{k=1}^n \mathfrak{e}_{I,k}\mathfrak{e}_{II,k} \end{cases} \to \begin{pmatrix} A_{1111} & A_{1112} & A_{1211} & A_{1212} \\ A_{1121} & A_{1122} & A_{1221} & A_{1222} \\ A_{2111} & A_{2112} & A_{2211} & A_{2212} \\ A_{2121} & A_{2122} & A_{2221} & A_{2222} \end{pmatrix}$$

solution of YBE$_g$ with $\epsilon_i \in \{-1, +1\}$ establishes bijection $\mathbb{C}^{1|1} \to \mathbb{C}^2$

$$R(u) = \begin{pmatrix} a_1(u) & 0 & 0 & \epsilon_1 d_1(u) \\ 0 & \epsilon_2 b_1(u) & c_1(u) & 0 \\ 0 & c_2(u) & \epsilon_2 b_2(u) & 0 \\ -\epsilon_1 d_2(u) & 0 & 0 & -a_2(u) \end{pmatrix} \to \begin{pmatrix} a_1(u) & 0 & 0 & d_1(u) \\ 0 & b_1(u) & c_1(u) & 0 \\ 0 & c_2(u) & b_2(u) & 0 \\ d_2(u) & 0 & 0 & a_2(u) \end{pmatrix}$$

An alternative to Kulish-Sklyanin proof on graded QYBE bijection.

$\mathfrak{su}(2) \times \mathfrak{su}(2)$ Hubbard type models

Hubbard model

One can try to generalise the prescription to higher dimensions. A wide class of models lies in **four-dimensional Hilbert space**, where site can be either vacant, occupied by a single fermion with spin up or down, or by a pair of fermions, *e.g.* Hubbard model

$$\mathbb{H}^{(Hub)} = \sum_i \sum_{\alpha=\uparrow,\downarrow} (c^\dagger_{\alpha,i} c_{\alpha,i+1} + c^\dagger_{\alpha,i+1} c_{\alpha,i}) + \mathfrak{u}\, n_{\uparrow,i} n_{\downarrow,i}$$

- The kinetic part is a **hopping term**, which allows neighboring-site dynamics
- Potential term measures the number of **fermionic pairs** on each site ($\mathfrak{u}$ sets the overall scale)

.

Hubbard Type Symmetry

1). We take models which have $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ symmetry
2). Models whose kinetic part is given by $\mathbb{H}_{kin}^{Hub}$.

The Hubbard model itself does not appear as one of the solutions, since its R-matrix has non-difference spectral dependence. [Shastry Class]

For **Hubbard type** models, one recovers spin chains with $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{su}(4), \mathfrak{su}(2|2), \mathfrak{sp}(4)$ and $\mathfrak{so}(4)$ symmetry algebras.

Additional **Twist**

In addition to integrable transforms, one now requires also a twist

$$\begin{cases} \tilde{R}_{\mathcal{V},\mathcal{W}} = (\mathcal{V} \otimes \mathcal{W})\, R\, (\mathcal{V} \otimes \mathcal{W})^{-1} \\ [R, \mathcal{V} \otimes \mathcal{V}] = [R, \mathcal{W} \otimes \mathcal{W}] = 0 \end{cases}$$

The two-particle representations for the Hubbard type Ansatz arise as pure or mixed pairs

2-particle modules

$\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariant Hamiltonian takes the form

$$
\begin{aligned}
\mathcal{H}|\phi_a\phi_b\rangle &= A|\phi_a\phi_b\rangle + B|\phi_b\phi_a\rangle + C\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\alpha\psi_\beta\rangle \\
\mathcal{H}|\phi_a\psi_\beta\rangle &= G|\phi_a\psi_\beta\rangle + H|\psi_\beta\phi_a\rangle \\
\mathcal{H}|\psi_\alpha\phi_b\rangle &= K|\psi_\alpha\phi_b\rangle + L|\phi_b\psi_\alpha\rangle \\
\mathcal{H}|\psi_\alpha\psi_\beta\rangle &= D|\psi_\alpha\psi_\beta\rangle + E|\psi_\beta\psi_\alpha\rangle + F\epsilon^{ab}\epsilon_{\alpha\beta}|\phi_a\phi_b\rangle
\end{aligned}
$$

$\phi_{1,2}$ and $\psi_{1,2}$ span the two independent $\mathfrak{su}(2)$ fundamental representations.

The most **general two-site operator** that commutes with both $\mathfrak{su}(2)$ oscillator representations contains ten parameters

$$\mathcal{H}_{12} = \sum_{\alpha\neq\beta}\Big[(c^\dagger_{\alpha,1}c_{\alpha,2} + c_{\alpha,1}c^\dagger_{\alpha,2})(C_1 + C_2(n_{\beta,1} - n_{\beta,2})^2) +$$

$$(c^\dagger_{\alpha,1}c_{\alpha,2} - c_{\alpha,1}c^\dagger_{\alpha,2})(C_3(n_{\beta,1} - \frac{1}{2}) + C_4(n_{\beta,2} - \frac{1}{2}))\Big]$$

$$+ (c^\dagger_{\uparrow,1}c^\dagger_{\downarrow,1}c_{\uparrow,2}c_{\downarrow,2} + c_{\uparrow,1}c_{\downarrow,1}c^\dagger_{\uparrow,2}c^\dagger_{\downarrow,2})C_5 + (c^\dagger_{\uparrow,1}c_{\downarrow,1}c^\dagger_{\downarrow,2}c_{\uparrow,2} + c^\dagger_{\downarrow,1}c_{\uparrow,1}c^\dagger_{\uparrow,2}c_{\downarrow,2})C_6$$

$$+ C_7(n_{\uparrow,1} - \frac{1}{2})(n_{\downarrow,1} - \frac{1}{2}) + C_8(n_{\uparrow,2} - \frac{1}{2})(n_{\downarrow,2} - \frac{1}{2})$$

$$+ C_9(n_{\uparrow,1} - n_{\downarrow,1})^2(n_{\uparrow,2} - n_{\downarrow,2})^2 +$$

$$+ (C_5 - C_6)(n_{\uparrow,1}n_{\downarrow,1} + n_{\uparrow,2}n_{\downarrow,2} - 1)(n_{\uparrow,1} - n_{\uparrow,2})(n_{\downarrow,1} - n_{\downarrow,2})$$

$$+ \frac{1}{2}C_5((n_{\uparrow,1} - n_{\downarrow,2})^2 + (n_{\downarrow,1} - n_{\uparrow,2})^2) + C_0,$$

$$C_0 = \frac{1}{2}(B + G + K),\ C_1 = \frac{1}{2}(L - H),\ C_2 = \frac{1}{2}(C - F + H - L),$$

$$C_3 = \frac{1}{2}(H + L - C - F),\ C_4 = \frac{1}{2}(C + F + H + L),\ C_5 = -B,\ C_6 = E,$$

$$C_7 = 2A + B - 2K,\ C_8 = 2A + B - 2G,\ C_9 = A + B + D + E - G - K.$$

Solution classes

Model	A	B	C	D	E	F	G	H	K	L
I	ρ	$-\rho$	0	0	0	0	a	$\rho e^{-\phi}$	$2\rho - a$	ρe^{ϕ}
II	0	0	0	ρ	ρ	0	a	$\rho e^{-\phi}$	$2\rho - a$	ρe^{ϕ}
III	ρ	$-\rho$	$\rho e^{-\phi}$	$-\rho$	ρ	$-\rho e^{\phi}$	0	0	0	0
IV	ρ	$-\rho$	$\rho e^{-\phi}$	ρ	$-\rho$	ρe^{ϕ}	0	0	0	0
V	$\frac{7}{4}\rho$	$-\rho$	$\frac{1}{2}\rho e^{-\phi}$	$\frac{7}{4}\rho$	$-\rho$	$\frac{1}{2}\rho e^{\phi}$	0	0	0	0
VI	0	0	0	a	0	0	b	0	c	0
VII	0	0	0	0	0	0	a	b	c	d
VIII	0	0	0	$a + c$	0	0	a	b	c	d
IX	ρ	$-\rho$	0	ρ	$-\rho$	0	a	$\rho e^{-\phi}$	$2\rho - a$	ρe^{ϕ}
X	ρ	$-\rho$	0	ρ	ρ	0	a	$\rho e^{-\phi}$	$2\rho - a$	ρe^{ϕ}
XI	ρ	$-\rho$	$\frac{1}{2}\rho e^{-\phi}$	ρ	$-\rho$	$\frac{1}{2}\rho e^{\phi}$	$\frac{3}{2}\rho$	$-\frac{3}{2}\rho$	$\frac{3}{2}\rho$	$-\frac{3}{2}\rho$
XII	0	0	$-\rho e^{-\phi}$	0	0	ρe^{ϕ}	0	ρ	0	$-\rho$

Table: Hubbard type $\mathcal{H}$-generators in the non-graded sector

Generalized Hubbard model

Hubbard prescription can be straightforwardly extended to include additional (*integrable*) interactions or potential deformations

$$\mathcal{K}_{Hub} = \sum_{\alpha=\uparrow,\downarrow} (c^\dagger_{\alpha,1} c_{\alpha,2} + c^\dagger_{\alpha,2} c_{\alpha,1}) \qquad \mathcal{H} = \mathcal{K}_{Hub} + \mathcal{K}_{pair} + \mathcal{K}_{flip} + V$$

$$\begin{aligned} \mathcal{K}_{pair} &= A_1 c^\dagger_{\uparrow,1} c^\dagger_{\downarrow,1} c_{\uparrow,2} c_{\downarrow,2} + A_2 c^\dagger_{\uparrow,2} c^\dagger_{\downarrow,2} c_{\uparrow,1} c_{\downarrow,1} \\ \mathcal{K}_{flip} &= A_3 c^\dagger_{\uparrow,1} c^\dagger_{\downarrow,2} c_{\downarrow,1} c_{\uparrow,2} + A_4 c^\dagger_{\downarrow,1} c^\dagger_{\uparrow,2} c_{\uparrow,1} c_{\downarrow,2} + A_5 c^\dagger_{\uparrow,1} c^\dagger_{\uparrow,2} c_{\downarrow,1} c_{\downarrow,2} \\ &+ A_6 c^\dagger_{\downarrow,1} c^\dagger_{\downarrow,2} c_{\uparrow,1} c_{\uparrow,2} \end{aligned}$$

$$\begin{aligned} V = \; & B_1 + B_2\, n_{\uparrow,1} + B_3\, n_{\downarrow,1} + B_4\, n_{\uparrow,1} n_{\downarrow,1} + \\ & B_5\, n_{\uparrow,2} + B_6\, n_{\uparrow,1} n_{\uparrow,2} + B_7\, n_{\downarrow,1} n_{\uparrow,2} + B_8\, n_{\uparrow,1} n_{\downarrow,1} n_{\uparrow,2} + \\ & B_9\, n_{\downarrow,2} + B_{10}\, n_{\uparrow,1} n_{\downarrow,2} + B_{11}\, n_{\downarrow,1} n_{\downarrow,2} + B_{12}\, n_{\uparrow,1} n_{\downarrow,1} n_{\downarrow,2} + \\ & B_{13}\, n_{\uparrow,2} n_{\downarrow,2} + B_{14}\, n_{\uparrow,1} n_{\uparrow,2} n_{\downarrow,2} + B_{15}\, n_{\downarrow,1} n_{\uparrow,2} n_{\downarrow,2} + B_{16}\, n_{\uparrow,1} n_{\downarrow,1} n_{\uparrow,2} n_{\downarrow,2} \end{aligned}$$

Integrable solutions in 4-dim

Applying boost procedure, one can find four integrable models

$$\mathcal{H}^{(15)} = \mathcal{K}_{Hub} + a_1(\mathrm{n}_{\uparrow,1} - \mathrm{n}_{\uparrow,2})^2 + a_2(\mathrm{n}_{\uparrow,1} - \mathrm{n}_{\uparrow,2}) + a_3(\mathrm{n}_{\downarrow,1} - \mathrm{n}_{\downarrow,2})^2 + a_4(\mathrm{n}_{\downarrow,1} - \mathrm{n}_{\downarrow,2})$$
$$\mathcal{H}^{(16)} = \mathcal{K}_{Hub} + a_1(\mathrm{n}_{\uparrow,1} - \mathrm{n}_{\uparrow,2})^2 + a_2(\mathrm{n}_{\uparrow,1} - \mathrm{n}_{\uparrow,2}) + a_3(\mathrm{n}_{\downarrow,1} + \mathrm{n}_{\downarrow,2}) + a_4(\mathrm{n}_{\downarrow,1} - \mathrm{n}_{\downarrow,2})$$
$$\mathcal{H}^{(17)} = \mathcal{K}_{Hub} + a_1(\mathrm{n}_{\uparrow,1} + \mathrm{n}_{\uparrow,2}) + a_2(\mathrm{n}_{\uparrow,1} - \mathrm{n}_{\uparrow,2}) + a_3(\mathrm{n}_{\downarrow,1} + \mathrm{n}_{\downarrow,2}) + a_4(\mathrm{n}_{\downarrow,1} - \mathrm{n}_{\downarrow,2})$$

There are no models with $\mathcal{K}_{pair} \neq 0$. A model with non-trivial spin flip and potential part

$$\mathcal{H}^{(18)} = \mathcal{K}_{Hub} + a\Big(\mathrm{c}^\dagger_{\uparrow,1}\mathrm{c}^\dagger_{\downarrow,2}\mathrm{c}_{\downarrow,1}\mathrm{c}_{\uparrow,2} + \mathrm{c}^\dagger_{\downarrow,1}\mathrm{c}^\dagger_{\uparrow,2}\mathrm{c}_{\uparrow,1}\mathrm{c}_{\downarrow,2} + \mathrm{c}^\dagger_{\uparrow,1}\mathrm{c}^\dagger_{\uparrow,2}\mathrm{c}_{\downarrow,1}\mathrm{c}_{\downarrow,2} + \mathrm{c}^\dagger_{\downarrow,1}\mathrm{c}^\dagger_{\downarrow,2}\mathrm{c}_{\uparrow,1}\mathrm{c}_{\uparrow,2}\Big)$$
$$+ (2a - b)(\mathrm{n}_{\uparrow,1} + \mathrm{n}_{\downarrow,1}) + b(\mathrm{n}_{\uparrow,2} + \mathrm{n}_{\downarrow,2}) - a(\mathrm{n}_{\uparrow,1} + \mathrm{n}_{\downarrow,1})(\mathrm{n}_{\uparrow,2} + \mathrm{n}_{\downarrow,2})$$

.

this model does not preserve spin orientation and is specific type of XYZ deformation of the Hubbard potential.

Complete YBE solution space

- Complete set of integrable $\mathbb{C}^2$-models found (*-magnets, Heisenberg, multivertex models)
- Novel multiparametric $\mathfrak{sl}_2$ sector, with associated deformed $\mathcal{Y}(\mathfrak{sl}_2)$ [de Leeuw, AP, Ryan '19], which includes 4 nontrivial families with up to **5 parameters**.
- In the $\mathbb{C}^4$ space we have found new models, that exhibit **fermion pair formation** and **generalised Hubbard type** models with most generic potential that are integrable [de Leeuw, AP, Retore, Ryan '19]. (*BA not applicable, a Quantum Spectral Curve for the latter is in progress*).

AdS/CFT Integrability

In particular AdS/CFT integrability implies agreement of global symmetries on both sides of the correspondence, e.g. $\mathcal{N} = 4$ **superconformal** symmetry and $AdS_5 \times S^5$ **superspace isometries** are described by covering supergroup $\widetilde{PSU}(2,2|4)$. It is based on $\mathfrak{psu}(2,2|4)$ Lie superalgebra of dimension 30|32 (even| odd).

Having 4|4-$\mathbb{C}$ supermatrices

$$\mathfrak{M} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \quad A, D \text{ even and } B, C \text{ odd } 4 \times 4\ \mathbb{C}\text{-blocks (NonGraßmann)}$$

with Lie bracket being graded supermatrix commutator $[\cdot,\cdot\}$

$$[\mathfrak{M},\mathfrak{R}\} = \mathfrak{M}\mathfrak{R} - (-1)^{\mathfrak{M}\mathfrak{R}}\mathfrak{R}\mathfrak{M} \quad S\mathrm{Tr}\,\mathfrak{M} = \mathrm{Tr}A - \mathrm{Tr}D \quad S\mathrm{Tr}\ [\mathfrak{M},\mathfrak{R}\} = 0$$

$$(-1)^{\mathfrak{M}\mathfrak{T}}\,[[\mathfrak{M},\mathfrak{R}\},\mathfrak{T}\} + (-1)^{\mathfrak{R}\mathfrak{M}}\,[[\mathfrak{R},\mathfrak{T}\},\mathfrak{M}\} + (-1)^{\mathfrak{T}\mathfrak{R}}\,[[\mathfrak{T},\mathfrak{M}\},\mathfrak{R}\} = 0$$

By appropriate supertrace restriction and centre projection 30|32-dim $\mathfrak{psl}(4|4,\mathbb{C})$ is obtained from $\mathfrak{gl}(4|4,\mathbb{C})$

Sigma models on coset superspaces

S^3 sigma model as $SU(2)$ PCM

$$S = -\frac{1}{2}\int d^2x \mathrm{Tr}\left[\mathcal{J}_+, \mathcal{J}_-\right] \qquad \mathcal{J} = g^{-1}dg \in \mathfrak{su}(2)$$

which under extensions could be generalised to supercoset model

$$\frac{\hat{\mathfrak{F}}}{\mathfrak{f}} = \frac{\hat{G}\times\hat{G}}{\mathfrak{f}} \qquad S_{MT} = \int d^2x \,\mathrm{STr}\left[\left(\mathcal{P}_+\mathcal{J}_+\right)\mathcal{J}_-\right] \text{ [Metsaev, Tseytlin '98]}$$

with bosonic diagonal subgroup $\mathfrak{f}$ of factorised supergroup $\hat{\mathfrak{F}} = \hat{G}\times\hat{G}$

$AdS_n \times S^n = \hat{G}/H$ supercosets, with superisometry $\hat{G}$ include:

- $AdS_5 \times S^5 \longrightarrow \dfrac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$
- $AdS_3 \times S^3 \longrightarrow \dfrac{PSU(1,1|2)\times PSU(1,1|2)}{SO(1,2)\times SO(3)}$
- $AdS_2 \times S^2 \longrightarrow \dfrac{PSU(1,1|2)}{SO(1,1)\times SO(2)}$

Automorphic non-difference integrability

Quantum integrability consistency

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)$$

where $R_{ij}(x_i,y_j) \neq R_{ij}(x_i - y_j)$, with transfer matrix provided accordingly

$$\mathcal{T}(u,\theta) = tr_0\left[R_{0L}(0,\theta_L)\ldots R_{01}(0,\theta_1)\right]$$

In present setting one can restrict to space with regular R and homogeneous limit of NN–spin-chain

$$\mathbb{Q}_2(\theta) = \sum_k \mathcal{H}_{k,k+1} \qquad \mathcal{H}(\theta) = P\frac{dR(u,\theta)}{du}\Big|_{u\to\theta} \qquad R_{ij}(u,u) = P_{ij}$$

Generically one can generate integrable hierarchy of commuting charges

$$\mathbb{Q}_{r+1} \simeq \frac{d^r log\left[\mathcal{T}(u,\theta)\right]}{du^r}\Big|_{u\to\theta} \qquad [\mathbb{Q}_r,\mathbb{Q}_s] = 0$$

Automorphic non-difference integrability

As first step, in the present setting it is possible to find generalised (extended) solution space from the commuting tower $\mathbb{Q}_r$, that will define set of algebraic constraints. It possible to proceed with transfer derivatives and use $R\mathcal{T}\mathcal{T}$-algebra, but instead it could accomplished by the **generating automorphism**

Generalised $\mathcal{B}$oost

$$\mathcal{B}\left[\mathbb{Q}_2\right] = \sum_{k=-\infty}^{+\infty} k\mathcal{H}_{k,k+1}(\theta) + \partial_\theta \qquad \mathbb{Q}_{r+1} = \left[\mathcal{B}\left[\mathbb{Q}_2\right], \mathbb{Q}_r\right] \quad r > 1$$

$$[\mathbb{Q}_{r+1}, \mathbb{Q}_2] \Rightarrow \left[[\mathcal{B}[\mathbb{Q}_2], \mathbb{Q}_r], \mathbb{Q}_2\right] + [d_\theta \mathbb{Q}_r, \mathbb{Q}_2] = 0$$

from here follows first order nonlinear ODE coupled system.

***R*- and *S*-matrices arising in string integrable backgrounds possess arbitrary spectral dependence. Is there a technique to find the underlying R-matrix?**

Constructing the R-matrix

Constraints

To obtain R-matrix, one can expand YBE to first order and associate spectral parameters, which will result in coupled differential system for R

$$\begin{cases} [R_{13}R_{23}, \mathcal{H}_{12}(u)] = (\partial_u R_{13})R_{23} - R_{13}(\partial_u R_{23}) & u_1 = u_2 \equiv u \\ [R_{13}R_{12}, \mathcal{H}_{23}(v)] = (\partial_v R_{13})R_{12} - R_{13}(\partial_v R_{12}) & u_2 = u_3 \equiv v \end{cases}$$

with $R_{ij} = R_{ij}(u, v)$ and equations are reduction from Sutherland equation.

Symmetries

- Norm and shift
- Reparameterised: $R(\mathfrak{f}(u), \mathfrak{f}(v))$ satisfies YBE
- Local Basis Transform:
  $R^{\mathcal{V}}(u, v) = [\mathcal{V}(u) \otimes \mathcal{V}(v)]\, R(u, v)\, [\mathcal{V}(u) \otimes \mathcal{V}(v)]^{-1}$
- Discrete Transform: PRP, R^T and PR^TP satisfy YBE from R.
- Twisted sector: for any two $\mathfrak{T}_1$ and $\mathfrak{T}_2$ with R-symmetries $[\mathfrak{T}_{1,2} \otimes \mathfrak{T}_{1,2}, R] = 0$, then $[\mathfrak{T}_1(u) \otimes \mathfrak{T}_2(v)]\, R\, [\mathfrak{T}_2(u) \otimes \mathfrak{T}_1(v)]^{-1}$

Gauge/Gravity Integrability

- AdS_3/CFT_2 defines $\boldsymbol{AdS_3} \times \boldsymbol{S}^3 \times \boldsymbol{\mathcal{M}}^4$ under two geometries that preserve 16 supercharges

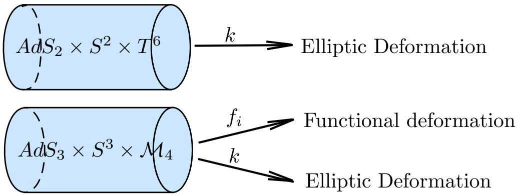
$$\begin{cases} \mathcal{M}^4 = T^4, \text{with } \mathfrak{psu}(1,1|2)^2 \\ \mathcal{M}^4 = S^3 \times S^1, \text{with } \mathfrak{d}(2,1;\alpha)^2 \sim \mathfrak{d}(2,1;\alpha)_L \oplus \mathfrak{d}(2,1;\alpha)_R \oplus \mathfrak{u}(1) \end{cases}$$

where α relates radii of the spheres.

- For $\boldsymbol{AdS_2} \times \boldsymbol{S}^2 \times \boldsymbol{T}^6$, $\mathfrak{psu}(1,1|2) \ni \mathbb{Z}_4$ automorphism, but no gauge choice for κ-symmetry

$AdS_{2,3}$ embedding

- For R-matrix of the $AdS_{2,3}$ consisting of different chirality 4×4 blocks, that satisfy qYBE.
- We find novel deformed Hamiltonians of $AdS_3 \times S^3 \times \mathcal{M}^4$ and $AdS_2 \times S^2 \times T^6$ type.
- AdS_3 admits either continuous family of deformations (**spectral functional shifts**) if mapped to **6-vB** or **single-parameter elliptic deformation** if mapped to **8-vB**.
- On the other hand, massive $AdS_2 \times S^2 \times T^6$ is of **8-vB** type and admits **single-parameter deformation**.



8v-A-B Classes

$$R^{8vA}(z) = \begin{pmatrix} \mathrm{sn}(\eta+z) & 0 & 0 & k\,\mathrm{sn}(\eta)\mathrm{sn}(z)\mathrm{sn}(\eta+z) \\ 0 & \mathrm{sn}(z) & \mathrm{sn}(\eta) & 0 \\ 0 & \mathrm{sn}(\eta) & \mathrm{sn}(z) & 0 \\ k\,\mathrm{sn}(\eta)\mathrm{sn}(z)\mathrm{sn}(\eta+z) & 0 & 0 & \mathrm{sn}(\eta+z) \end{pmatrix}$$

8-vertex B class

$$r_1 = \Sigma(u,v)\left[\sin\eta_+\frac{\mathrm{cn}}{\mathrm{dn}} - \cos\eta_+\mathrm{sn}\right]$$

$$r_2 = -\Sigma(u,v)\left[\cos\eta_-\mathrm{sn} + \sin\eta_-\frac{\mathrm{cn}}{\mathrm{dn}}\right]$$

$$r_3 = -\Sigma(u,v)\left[\cos\eta_-\mathrm{sn} - \sin\eta_-\frac{\mathrm{cn}}{\mathrm{dn}}\right]$$

$$r_4 = \Sigma(u,v)\left[\sin\eta_+\frac{\mathrm{cn}}{\mathrm{dn}} + \cos\eta_+\mathrm{sn}\right]$$

$$r_5 = r_6 = 1, \qquad r_7 = r_8 = k\,\mathrm{sn}\frac{\mathrm{cn}}{\mathrm{dn}}$$

with elliptic functions to be $\mathrm{xn} = \mathrm{xn}(u-v,k^2)$, $\Sigma(u,v) = [\sin\eta(u)\sin\eta(v)]^{-\frac{1}{2}}$, $\eta_\pm \equiv \frac{\eta(u)-\eta(v)}{2}$ for arbitrary function $\eta(u)$ and constant k.

The four block implementation will result in the R operator of the form

$$\begin{pmatrix}
r_1^{LL} & 0 & 0 & 0 & 0 & r_8^{LL} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r_2^{LL} & 0 & 0 & r_6^{LL} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & r_1^{LR} & 0 & 0 & 0 & 0 & r_8^{LR} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_2^{LR} & 0 & 0 & r_6^{LR} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r_5^{LL} & 0 & 0 & r_3^{LL} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
r_7^{LL} & 0 & 0 & 0 & 0 & r_4^{LL} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & r_5^{LR} & 0 & 0 & r_3^{LR} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & r_7^{LR} & 0 & 0 & 0 & 0 & r_4^{LR} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1^{RL} & 0 & 0 & 0 & 0 & r_8^{RL} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_2^{RL} & 0 & 0 & r_6^{RL} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1^{RR} & 0 & 0 & 0 & 0 & r_8^{RR} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_2^{RR} & 0 & 0 & r_6^{RR} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_5^{RL} & 0 & 0 & r_3^{RL} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7^{RL} & 0 & 0 & 0 & 0 & r_4^{RL} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_5^{RR} & 0 & 0 & r_3^{RR} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_7^{RR} & 0 & 0 & 0 & 0 & r_4^{RR}
\end{pmatrix}$$

where $r_k^{\mathcal{X}} \equiv r_k^{\mathcal{X}}(u, v)$, $\mathcal{X} \in \{\text{LL, RR, LR, RL}\}$ and R will correspond to the full 16×16 R-matrix (if not stated otherwise).

Structure and Properties: $AdS_{\{2,3\}}$ deformed Limits

Reductions

- Important that $AdS_3 \times S^3 \times S^3 \times S^1$ can be obtained from 6-vertex B (trigonometric) by appropriate parametric identification in Zhukovsky space (one can build $AdS_3 \times S^3 \times T^4$ R-/S-matrix from $AdS_3 \times S^3 \times S^3 \times S^1$ with $\alpha \to \{0, 1\}$ limits).
- As well as 8-vertex model B (elliptic), which is a deformation of $AdS_2 \times S^2 \times T^6$.
- Moreover two-parameter q-deformed R-matrix that underlies double deformed σ-model can be embedded into 6vB model.

AdS_3 2-parameter deformation

It is possible to construct two-parameter deformation of $AdS_3 \times S^3 \times \mathcal{M}^4$ backgrounds, by deforming each copy of the factorised supergroup

$$S = \int d^2x \,\mathrm{STr}\left[\mathcal{J}_+ \left(\mathcal{P}_-^{\eta_{L,R}} \frac{1}{1 - I_{\eta_{L,R}} R_f \mathcal{P}_-^{\eta_{L,R}}} \mathcal{J}_-\right)\right] \quad \text{[Hoare '14]}$$

$$R_f = \mathrm{Ad}_f^{-1} R \mathrm{Ad}_f \quad I_{\eta_{L,R}} = \frac{2}{\sqrt{\left(1-\eta_L^2\right)\left(1-\eta_R^2\right)}} \begin{pmatrix} \eta_L \mathbb{1} & 0 \\ 0 & \eta_R \mathbb{1} \end{pmatrix}$$

$$\mathcal{P}_\pm^{\eta_{L,R}} = P_2 \mp \frac{\sqrt{\left(1-\eta_L^2\right)\left(1-\eta_R^2\right)}}{2} \left(\mathcal{P}_1 - \mathcal{P}_3\right)$$

with single-parameter $\eta_{L,R} = \eta$ and undeformed$\eta_{L,R} \to 0$ case. The fundamental R-matrix defined on $\mathcal{U}_q\left(\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3\right)$ is entirely fixed by co-commutativity with the coproduct

$$\Delta^{op}(\mathfrak{J}) R = R \Delta(\mathfrak{J}) \quad \Delta^{op}(\mathfrak{J}) = \mathcal{P} \Delta(\mathfrak{J})$$

Double q-deformed algebra and representations

$$[\mathfrak{B},\mathfrak{O}_\pm] = \pm 2i\mathfrak{O}_\pm \qquad [\mathfrak{B},\mathfrak{S}_\pm] = \pm 2i\mathfrak{S}_\pm$$

$$\{\mathfrak{O}_+,\mathfrak{S}_-\} = \mathfrak{C}+\mathfrak{M} = \mathfrak{C}_L \qquad \{\mathfrak{O}_-,\mathfrak{S}_+\} = \mathfrak{C}-\mathfrak{M} = \mathfrak{C}_R$$

$$\{\mathfrak{O}_+,\mathfrak{O}_-\} = \mathfrak{P} \qquad \{\mathfrak{S}_+,\mathfrak{S}_-\} = \mathfrak{K}$$

with $\mathfrak{B}$ an automorphism of $\mathfrak{u}(1)$, supercharges $\mathfrak{O}_\pm$, $\mathfrak{S}_\pm$, and central elements $\mathfrak{M}$, $\mathfrak{C}$, $\mathfrak{P}$, $\mathfrak{K}$. For that one to deform central elements of the superalgebras separately:

$$\{\mathfrak{O}_\alpha,\mathfrak{S}_\beta\} = [\mathfrak{C}_I]_{q_I} = \frac{\mathfrak{V}_I - \mathfrak{V}_I^{-1}}{q_I - q_I^{-1}},\ \mathfrak{V}_I = q_I^{\mathfrak{C}_I} \quad \alpha = \pm, \beta = \mp, I = L/R$$

Corproduct structure is defined through generator action on tensor product representations

$$\Delta(\mathfrak{O}_+) = \mathfrak{O}_+ \otimes \mathbb{1} + \mathfrak{U}\mathfrak{V}_L \otimes \mathfrak{O}_+ \qquad \Delta(\mathfrak{O}_-) = \mathfrak{O}_- \otimes \mathbb{1} + \mathfrak{U}\mathfrak{V}_R \otimes \mathfrak{O}_-$$

$$\Delta(\mathfrak{S}_+) = \mathfrak{S}_\pm \otimes \mathfrak{V}_R^{-1} + \mathfrak{U}^{-1} \otimes \mathfrak{S}_\pm \qquad \Delta(\mathfrak{S}_-) = \mathfrak{S}_\pm \otimes \mathfrak{V}_L^{-1} + \mathfrak{U}^{-1} \otimes \mathfrak{S}_\pm$$

$$\Delta(\mathfrak{P}) = \mathfrak{P} \otimes \mathbb{1} + \mathfrak{U}^2\mathfrak{V}_L\mathfrak{V}_R \otimes \mathfrak{P} \qquad \Delta(\mathfrak{K}) = \mathfrak{K} \otimes \mathfrak{V}_L^{-1}\mathfrak{V}_R^{-1} + \mathfrak{U}^{-2} \otimes \mathfrak{K}$$

Free Fermion Condition

Classes

These two classes could identified by the R algebraic condition

$$\frac{\left[r_1 r_4 + r_2 r_3 - (r_5 r_6 + r_7 r_8)\right]^2}{r_1 r_2 r_3 r_4} = \mathfrak{c}_\mathrm{B}$$

where $\mathfrak{c}_\mathrm{B}$ constitutes a characteristic Baxter constant with

$$\begin{cases} \mathfrak{c}_\mathrm{B} = 0, & \textit{Free Fermion} \text{ constraint} \qquad [B] \\ \mathfrak{c}_\mathrm{B} \neq 0, & \textit{Baxter} \text{ constraint} \qquad [A] \end{cases}$$

For AdS_3 with RR, the massless R-matrix is described by nested BA, where pseudovacuum consisting of $|\phi\rangle$ is level-one pseudovacuum and not the corresponding BMN vacuum of all $|Z\rangle$ [Ohlsson Sax et. al. '12], so that for the transfer matrix

$$t_N = \mathrm{str}_0 R_{01}(\theta_0 - \theta_1) \ldots R_{0N}(\theta_0 - \theta_N) \qquad (1)$$

FF: Pure RR flux

For AdS_3 with pure Ramond-Ramond, the massless R-matrix is described by nested BA, where pseudovacuum consisting of $|\phi\rangle$ is level-one pseudovacuum and not the corresponding BMN vacuum of all $|Z\rangle$ [Ohlsson Sax et. al. '12], so that for the transfer matrix

$$t_N = \mathrm{str}_0 R_{01}(\theta_0 - \theta_1)...R_{0N}(\theta_0 - \theta_N)$$

$$\begin{aligned} t_2 &= \frac{1 - b_{01} b_{02}}{a_{01} a_{02}} (m_1 m_2 - n_1 n_2) + \frac{b_{01} - b_{02}}{a_{01} a_{02}} (m_1 n_2 - n_1 m_2) + c_1^\dagger c_2 - c_1 c_2^\dagger \\ &= \frac{1}{a_{12}} \mathbb{1} - e^{-\frac{\theta_{12}}{2}} c_1^\dagger c_1 - e^{\frac{\theta_{12}}{2}} c_2^\dagger c_2 + c_1^\dagger c_2 - c_1 c_2^\dagger \end{aligned}$$

$$\begin{cases} c_1 = \cos\alpha\,\eta_1 - \sin\alpha\,\eta_2 \\ c_2 = \sin\alpha\,\eta_1 + \cos\alpha\,\eta_2 \\ \cot 2\alpha = \sinh\frac{\theta_{12}}{2} \in \mathbb{R} \end{cases}$$

FF: Mixed flux

In the massless RR-NSNS flux case one can acquire the transformations

$$t_2^{\text{RR-NS}} = \mathfrak{a} + \mathfrak{b}\mathbb{N}_1 + (\mathfrak{b}-2)\mathbb{N}_2 + \mathfrak{c}\mathbb{N}_1\mathbb{N}_2$$

with

$$\mathfrak{a} = \frac{e^{-\frac{1}{2}(2\theta_0+\theta_1+\theta_2)}\left(e^{2i\frac{\pi}{k}+2\theta_0}-e^{\theta_1+\theta_2}\right)}{e^{2i\frac{\pi}{k}}-1}$$

$$\mathfrak{b} = \frac{1+e^{i\frac{\pi}{k}}-e^{i\pi\frac{\pi}{k}+\theta_0-\frac{\theta_1}{2}-\frac{\theta_2}{2}}-e^{\frac{1}{2}(-2\theta_0+\theta_1+\theta_2)}}{1+e^{i\frac{\pi}{k}}}$$

$$\mathfrak{c} = 2i\sinh\left(\theta_0-\frac{\theta_1}{2}-\frac{\theta_2}{2}\right)\tan\frac{\pi}{2k}$$

FF: Massive AdS_3

$$R_{\mathrm{m}AdS_3} = \mathfrak{A}\, E_{11} \otimes E_{11} + \mathfrak{B} E_{11} \otimes E_{22} + \mathfrak{C} E_{21} \otimes E_{12} - \mathfrak{F} E_{22} \otimes E_{22} + \mathfrak{G} E_{22} \otimes E_{11} - \mathfrak{H} E_{12} \otimes E_{21},$$

it can be established that the R-matrix functions satisfy

$$\mathfrak{A}\mathfrak{F} + \mathfrak{B}\mathfrak{G} = \mathfrak{C}^2 \qquad \mathfrak{C} = \mathfrak{H} \tag{2}$$

which can be demonstrated to generate FF condition for the massive case. In the FF reduced form it appears analogous to massless case, albeit distinct α-parametrisation

$$\tan 2\alpha = \frac{2\mathfrak{H}}{\mathfrak{G} - \mathfrak{B}} = -2 \left(\frac{x_p^-}{x_p^+} \frac{x_q^+}{x_q^-} \right)^{\frac{1}{4}} \frac{\sqrt{x_p^- - x_p^+} \sqrt{x_q^- - x_q^+}}{\sqrt{\frac{x_p^-}{x_p^+}}(x_p^+ - x_q^+) - \sqrt{\frac{x_q^+}{x_q^-}}(x_p^- - x_q^-)}$$

This parametrisation takes finite non-trivial value in the BMN limit ($\mathbb{R}$ in the physical neighbourhood).

Braiding properties constitute an important operator integrability characteristic and follow from the braiding unitarity constraint

Braiding unitarity

$$R^{\mathcal{X}} P \bar{R}^{\bar{\mathcal{X}}} P = \mathfrak{B}^{\mathcal{X}} \mathbb{1}$$

where $R \equiv R(u,v)$, $\mathfrak{B} \equiv \mathfrak{B}(u,v)$, the chiral sector $\mathcal{X}$ and bar implies swap of spectral parameters and chiralities (only mixed sectors affected).

$$\mathfrak{B}^{\mathsf{LL}} = \frac{h_2^{\mathsf{L}}(u)-h_1^{\mathsf{L}}(v)}{h_2^{\mathsf{L}}(u)-h_1^{\mathsf{L}}(u)} \frac{h_2^{\mathsf{L}}(v)-h_1^{\mathsf{L}}(u)}{h_2^{\mathsf{L}}(v)-h_1^{\mathsf{L}}(v)} \sigma^{\mathsf{LL}}(u,v)\sigma^{\mathsf{LL}}(v,u)$$

$$\mathfrak{B}^{\mathsf{RR}} = \frac{h_2^{\mathsf{R}}(u)-h_1^{\mathsf{R}}(v)}{h_2^{\mathsf{R}}(u)-h_1^{\mathsf{R}}(u)} \frac{h_2^{\mathsf{R}}(v)-h_1^{\mathsf{R}}(u)}{h_2^{\mathsf{R}}(v)-h_1^{\mathsf{R}}(v)} \sigma^{\mathsf{RR}}(u,v)\sigma^{\mathsf{RR}}(v,u)$$

$$\mathfrak{B}^{\mathsf{LR}} = \frac{1+h_2^{\mathsf{L}}(u)h_2^{\mathsf{R}}(v)}{1+h_1^{\mathsf{L}}(u)h_2^{\mathsf{R}}(v)} \frac{1+h_1^{\mathsf{R}}(v)h_1^{\mathsf{L}}(u)}{1+h_1^{\mathsf{R}}(v)h_2^{\mathsf{L}}(u)} \sigma^{\mathsf{LR}}(u,v)\sigma^{\mathsf{RL}}(v,u)$$

$$\mathfrak{B}^{\mathsf{RL}} = \frac{1+h_2^{\mathsf{L}}(v)h_2^{\mathsf{R}}(u)}{1+h_1^{\mathsf{L}}(v)h_2^{\mathsf{R}}(u)} \frac{1+h_1^{\mathsf{R}}(v)h_1^{\mathsf{L}}(u)}{1+h_1^{\mathsf{R}}(u)h_2^{\mathsf{L}}(v)} \sigma^{\mathsf{RL}}(u,v)\sigma^{\mathsf{LR}}(v,u)$$

For the full R embedding

$$R(u,v)PR(v,u)P = B(u,v)\mathbb{1} \qquad \text{iff} \quad \mathfrak{B}^{\{\mathsf{LL},\mathsf{RR},\mathsf{LR},\mathsf{RL}\}} = B$$

Crossing symmetry: 6vB

- Generically individual blocks obey crossing symmetry and braiding unitarity. It is important to resolve if **full scattering operator** does.
- The crossing symmetry works for $AdS_{\{2,3\}}$ bosofermionic R for generic k, although conjugation operator of AdS_2 require s further analysis.

The 6vB AdS_3 deformation R-matrix satisfies crossing through

$$\mathbb{C}_i R(u+\Delta_{\omega,1}, v+\Delta_{\omega,2})^{t_i}\mathbb{C}_i^{-1} = R(u,v)^{-1} \qquad \begin{cases} i=1, & \Delta_{\omega,1}=\omega,\ \Delta_{\omega,2}=0 \\ i=2, & \Delta_{\omega,1}=0,\ \Delta_{\omega,2}=-\omega \end{cases}$$

$$\mathbb{C}^{\text{6vB}}_{AdS_3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

where by i we identify the corresponding vector space, t_i are transpositions in space i and ω is a crossing parameter.

6vB Crossing constraining

The above appears to hold under

- Constraint on $h_i^{\mathsf{L}/\mathsf{R}}(u \pm \omega)$ with $i = 1, 2$

$$h_i^{\mathsf{R}}(u \pm \omega) = -\frac{1}{h_i^{\mathsf{L}}(u)}, \qquad h_i^{\mathsf{L}}(u \pm \omega) = -\frac{1}{h_i^{\mathsf{R}}(u)} \tag{3}$$

  which implies

$$h_i^{\mathsf{x}}(u) = h_i^{\mathsf{x}}(u \pm 2\omega) \tag{4}$$

- Constraint on $\mathfrak{X}$ and $\mathfrak{Y}$

$$\begin{aligned} \mathfrak{X}^{\mathrm{x}_1}(u \pm 2\omega) &= -\mathfrak{X}^{\mathrm{x}_1}(u) & \qquad \mathfrak{X}^{\mathsf{R}}(u) &= \mathfrak{X}^{\mathsf{L}}(u + \omega) \\ \mathfrak{Y}^{\mathrm{x}_1}(u \pm 2\omega) &= -\mathfrak{Y}^{\mathrm{x}_1}(u) & \qquad \mathfrak{Y}^{\mathsf{R}}(u) &= \mathfrak{Y}^{\mathsf{L}}(u + \omega) \end{aligned} \tag{5}$$

6vB Crossing constraining

- Constraining the scalar σ-factors

$$\sigma^{\mathrm{x}_2\mathrm{x}_1}(u, v-\omega) = \sigma^{\mathrm{x}_1\mathrm{x}_2}(u+\omega, v)$$

$$\sigma^{\mathrm{x}_1\mathrm{x}_1}(u, v-\omega) = -h_2^{\mathrm{x}_1}(u) h_2^{\mathrm{x}_2}(v) \sigma^{\mathrm{x}_2\mathrm{x}_2}(u+\omega, v)$$

$$\sigma^{\mathrm{x}_1\mathrm{x}_2}(u+\omega, v)\sigma^{\mathrm{x}_2\mathrm{x}_2}(u, v) = \frac{h_2^{\mathrm{x}_2}(u)-h_1^{\mathrm{x}_2}(u)}{h_2^{\mathrm{x}_2}(v)-h_1^{\mathrm{x}_2}(u)}$$

$$\sigma^{\mathrm{x}_1\mathrm{x}_1}(u+\omega, v)\sigma^{\mathrm{x}_2\mathrm{x}_1}(u, v) = \frac{h_2^{\mathrm{x}_2}(u)-h_1^{\mathrm{x}_2}(u)}{h_2^{\mathrm{x}_2}(u)} \frac{1+h_1^{\mathrm{x}_1}(v)h_2^{\mathrm{x}_2}(u)}{\left(1+h_1^{\mathrm{x}_2}(u)h_1^{\mathrm{x}_1}(v)\right)\left(1+h_2^{\mathrm{x}_2}(u)h_2^{\mathrm{x}_1}(v)\right)}$$

where $\mathrm{x}_k = \{\mathrm{L}, \mathrm{R}\}$ denotes appropriate chirality with $k = 1, 2$ and $\mathrm{x}_1 \neq \mathrm{x}_2$.

Crossing symmetry: 8vB

In the 8vB AdS_2 case one derives

$$\mathbb{C}_i R(u+\Delta_{\omega,1}, v+\Delta_{\omega,2})^{st_i}\mathbb{C}_i^{-1} = R(u,v)^{-1} \qquad \begin{cases} i=1, & \Delta_{\omega,1}=\omega,\ \Delta_{\omega,2}=0 \\ i=2, & \Delta_{\omega,1}=0,\ \Delta_{\omega,2}=-\omega \end{cases}$$

since the R-matrix is in the bosofermionic form the super-transposition applies in the i-space and conjugation operator is obtained

$$\mathbb{C}^{\text{8vB}}_{AdS_2} = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}$$

Despite that it is a deformation of the AdS_2 model, the conjugation operator is different from the one studied before (mapping to anti-particles). For the present case it is of super-type (boson $\leftrightarrow$ fermion).

Crossing constraining: 8vB

The 8vB bosofermion R-matrix internal functions must satisfy

$$\eta(u+\omega) = -\eta(u) + 2\pi n \qquad \mathcal{F}(u+\omega) = \mathcal{F}(u) + 2\,n\,K$$

$$\eta(u-\omega) = -\eta(u) + 2\pi m \qquad \mathcal{F}(u-\omega) = \mathcal{F}(u) + 2\,m\,K$$

where $m, n \in \mathbb{Z}$ and the 1st kind elliptic integral $K(k^2)$. For the dressing factors one obtains

$$\sigma(u+\omega, v)\sigma(u,v) = \sigma(u, v-\omega)\sigma(u,v) = i\left[(\mathrm{sn}\,\Sigma\cos\eta_-)^2 - \left(\frac{\mathrm{cn}}{\mathrm{dn}}\Sigma\sin\eta_-\right)^2 - 1\right]^{-1}$$

For the AdS_2 deformation the **boson-boson** R-matrix does satisfy crossing symmetry $\forall k\backslash\{k\to 1\}$, with conditionals of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

along with

$$\eta(u+\omega) = -\eta(u) + \pi n \qquad \mathcal{F}(u+\omega) = \mathcal{F}(u) + \,n\,K$$

$$\eta(u-\omega) = -\eta(u) - \pi m \qquad \mathcal{F}(u-\omega) = \mathcal{F}(u) + \,m\,K$$

where m, n hold to be odd.

Conclusions

- We have **constructed a method based on automorphic symmetries**, which appeared universal for many classes of integrable systems on the lattice, *e.g.* regardless dimension, symmetry, spectral dependence and other.
- We proposed **generating automorphsm for non-difference form** models arising in AdS_n integrable backgrounds.
- We developed ***R*-matrix construction** approach for string type setups and identified a set of invariant integrable transformations.
- It was shown that string type class **B** exhibits **free-fermion**, **braiding-unitarity**, **conjugation** and other properties.
- We have identified the properties and structure of the found $AdS_{\{2,3\}}$ deformed models as their **6vB**/**8vB** realisation.

Further directions

- **Wrapping formalism** for $AdS_3 \times S^3 \times T^4$ RR and **GSE** derivation (deformed Lüscher formulation) [Frolov, AP *to appear soon*]
- To develop Generalised Algebraic Bethe Ansatz [Slavnov, Zabrodin, Zotov '20] with associated graded criterion (supersymmetric selection) for AdS_2 and its deformation (that also can be used to construct deformed algebra).
- Notion of generalised flux and control parameter in non-difference vertex models? Connection to RR-NSNS case?
- $\mathfrak{sl}_2$ sector provides other deformations: Do they relate to ADHR in a limit, other models? Is there interpretation in terms of inhomogeneous spin chains as of [Dedushenko, Gaiotto '20]?
- Quantum algebras and $\mathcal{Y}_n$? Is there quantum cohomological classification? Belavin-Drinfeld classification?
- AdS_5 sector remains under consideration, but AdS_5 restricted ansatz might not provide more that deformation of the Hubbard chain.

- Can one consider full $\mathcal{PT}$-invariant model classification and resolution?
- Are there Bethe Ansätze that would solve Generalised Hubbard type models or if Quantum Spectral Curve can be devloped for such models?
- Can this tell us more about generalised multi-layer symmetries in AdS_5 (as of [Mitev, Staudacher, Tsuboi '12], [Shiroishi, Wadati '95])? Should one consider generic ansatz based on Korepanov construction?
- Present **6vB** does not admit 3-parametric deformation by appropriate spectral restrictions. However with procedure defined one can address construction of the scattering operator (not known even at algebraic level)? Relate it to Lukyanov 4-parametric NLSM? Limits, reductions in/to harmonic map problem and its symmetries?
- There also should be a possibility to restrict (spectrally) 6vB and 8vB class in order to develop full underlying deformed superalgebra, including RTT or Quantum Inverse Scattering Scattering.